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Universality and the five-dimensional Ising model

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Abstract. – We solve the long-standing discrepancy between Monte Carlo results and the renormalization prediction for the Binder cumulant of the five-dimensional Ising model. Our conclusions are based on accurate Monte Carlo data for systems with linear sizes up to $L = 22$. A detailed analysis of the corrections to scaling allows the extrapolation of these results to $L = \infty$. Our determination of the critical point, $K_c = 0.1139150$ (4), is more than an order of magnitude more accurate than previous estimates.

Introduction. – A much-debated issue in recent years is the question of universality of the five-dimensional Ising model [1]-[8]. This question focuses on the value of the renormalized coupling constant g at criticality, which is related to the Binder cumulant B [9]. For d-dimensional systems with periodic boundary conditions, hypercubic geometry and $d \geq d_u$, where d_u denotes the upper critical dimension, renormalization theory [10] predicts that this cumulant has a universal value. However, although $d_u = 4$ for Ising models with short-range couplings, Monte Carlo simulations [1], [2] for five-dimensional Ising systems with linear sizes $3 \leq L \leq 7$ suggested a different value for B. Large-scale simulations for $5 \leq L \leq 17$ [3] corroborated the earlier Monte Carlo result. As this controversy might indicate a problem with the renormalization analysis, various efforts were undertaken to gain additional insight in the nature of the discrepancy. First, simulations were carried out for a closely related class of systems, namely low-dimensional systems with algebraically decaying interactions [4]. Provided that these long-range interactions decay sufficiently slowly, they induce classical critical behaviour even in one-dimensional systems. Thus they effectively lower the upper critical dimension. As these systems are described by the same renormalization equations as high-dimensional short-range models, the same discrepancy in the Binder cumulant could be observable. The advantage of examining these long-range systems is their lower dimensionality, which makes it possible to simulate a much larger range of system sizes. It might seem that this advantage is undone by the increase of simulation time due to the larger number of interacting neighbours, which constitutes the very reason for the mean-field–like behaviour. However, this latter problem was avoided by a cluster algorithm for long-range interactions in which the simulation time is independent of the number of interacting spins [11]. The Binder cumulant was shown to agree accurately with the theoretical prediction for all examined systems with $d > d_\mathrm{u}$ (for $d = 1, 2, 3$). Nevertheless, this did not completely resolve the existing discrepancy, as the relation between models with long-range interactions and high-dimensional short-range Ising models is non-exact. Two subsequent studies actually were concerned with the five-dimensional model itself. Mon [5] studied the finite-size behaviour of the first and third absolute magnetization moments (normalized by the second moment to render them dimensionless), $\langle|m|\rangle/\langle m^2\rangle^{1/2}$ and $\langle|m^3|\rangle/\langle m^2\rangle^{3/2}$, and found that the Monte Carlo results for these quantities agreed well with the theoretically expected values. Furthermore, he showed that the finite-size corrections for the fourth moment —which is directly related to the Binder cumulant— are much larger than for the first and third one, which might explain the previously found disagreement. The only point of discussion concerning this study was the nature of the dominant finite-size correction, see refs. [7], [8]. Next, Parisi and Ruiz-Lorenzo [6] carried out Monte Carlo simulations for the five-dimensional Ising model using the Wolff cluster algorithm, which implied a considerable improvement compared to previous studies. They also introduced a new quantity, namely the Binder cumulant evaluated at the "apparent critical temperature", defined as the (size-dependent) temperature where the connected susceptibility takes its maximum. They showed their Monte Carlo results for this quantity, taken at $4 \leq L \leq 16$, to agree well with the mean-field value. Unfortunately, the statistical accuracy of the numerical results for the Binder cumulant at the critical temperature was not sufficient to allow an extrapolation to the $L \to \infty$ limit, so that the original controversy could not be settled yet.

Simulations. – In this paper we present new Monte Carlo results for the five-dimensional Ising model. We have carried out simulations for hypercubic systems up to linear size $L = 22$, which corresponds to more than 5×10^6 spins. Periodic boundaries were employed. The results have a high statistical accuracy, which is required to resolve the various finite-size corrections. The majority of the results were obtained on a Cray T3E massively parallel computer at Delft University. A total amount of 4000 (one-processor) CPU-hours was invested. One quarter of the total time was spent on the two largest system sizes, $L = 20, 22$. As in ref. [6], we used the Wolff cluster algorithm [12] to suppress critical slowing-down. Samples were taken at intervals containing a number of Wolff steps approximately equal to the inverse of the average relative cluster size. Table I gives the details for the various system sizes.

Results and discussion. – The main quantity of interest is the universal $L \to \infty$ limit of the amplitude ratio

$$Q(T, L) \equiv \frac{\langle m_L^2\rangle^2}{\langle m_L^4\rangle}\ , \tag{1}$$

which is directly related to the Binder cumulant $B = -3 + 1/Q$. In order to analyze the finite-size data for $Q(T, L)$ we need a description of the corrections to scaling. Above the upper critical dimension, the theory of scaling differs from that below d_u, because of the presence of a so-called dangerous irrelevant variable. This leads to a violation of hyperscaling. A detailed discussion of the form of the finite-size scaling functions is given in ref. [4]. The resulting prediction of renormalization theory is

$$Q(T, L) = \tilde{Q}\left(\hat{t}L^{y_\mathrm{t}^*}, uL^{y_\mathrm{i}}\right) + b_1 L^{d-2y_\mathrm{h}^*} + \cdots\ , \tag{2}$$

where $\tilde{Q}$ is a *universal* function, $\hat{t} = t + \alpha L^{y_\mathrm{i}-y_\mathrm{t}}$ and uL^{y_i} originates from irrelevant higher-order contributions in the renormalization equations [4]. $t \equiv (T - T_\mathrm{c})/T_\mathrm{c}$ is the reduced temperature.

TABLE I. – *Details of the Monte Carlo simulations. The table shows both the number of Wolff clusters per sample and the total number of samples taken for each system size.*

System size	Clusters/sample	Million samples
2	5	40
3	10	36
4	20	21
5	30	13
6	50	13
7	70	5.3
8	100	5.8
9	120	3.0
10	200	2.7
11	200	1.6
12	250	1.9
13	320	0.77
14	400	0.95
15	500	0.51
16	600	0.64
17	700	0.38
18	800	0.32
19	900	0.29
20	1000	0.26
22	1400	0.19

The asterisks indicate that the exponents are modified by the dangerous irrelevant variable. Following the notation of ref. [4] we have $y_{\mathrm{t}}^* = y_{\mathrm{t}} - y_{\mathrm{i}}/2$ and $y_{\mathrm{h}}^* = y_{\mathrm{h}} - y_{\mathrm{i}}/4$, where in turn $y_{\mathrm{t}} = 2$ is the thermal exponent, $y_{\mathrm{h}} = (2+d)/2$ the magnetic exponent and $y_{\mathrm{i}} = 4-d$ the leading irrelevant exponent. Thus, $y_{\mathrm{t}}^* = d/2$ and $y_{\mathrm{h}}^* = 3d/4$. The one-loop correction αL^{2-d} in $\hat{t}$ is the so-called shift in the critical temperature, which leads to a finite-size correction proportional to $L^{y_{\mathrm{i}}/2} = L^{2-d/2}$ in $Q(T,L)$. The term $b_1 L^{-d/2}$ arises from the analytic part of the free energy and the ellipsis stands for higher-order terms. Upon expansion of the scaling formula for $Q(T,L)$ near criticality, one finds:

$$Q(T,L) = Q + a_1 \hat{t} L^{y_{\mathrm{t}}^*} + a_2 \hat{t} L^{2y_{\mathrm{t}}^*} + \cdots + b_1 L^{d-2y_{\mathrm{h}}^*} + \cdots + c_1 L^{y_{\mathrm{i}}} + \cdots . \qquad (3)$$

We have fitted eq. (3) to our finite-size data. All data for $L \geq 5$ were included in the analysis. In addition to the terms in (3) we also used one cross-term in the expansion, viz. $\hat{t} L^{y_{\mathrm{t}}^* + y_{\mathrm{i}}}$. The exponents of the correction terms, y_{i} and $d - 2y_{\mathrm{h}}^*$, were kept fixed. The results are shown in table II. In the first analysis, one observes that both y_{t}^* and Q agree with the theoretical predictions, $y_{\mathrm{t}}^* = d/2$ and $Q = 8\pi^2/[\Gamma(\frac{1}{4})]^4 \approx 0.456947$. Comparing this to previous studies, we make the following remarks. The best estimate in ref. [3] is $Q = 0.489$ (6), more than five standard deviations from the renormalization prediction. This value deviates approximately four (combined) standard deviations from our result. Furthermore, the quoted error margin is of the same order as ours. Since our data have much smaller statistical errors, this indicates that less correction terms were taken into account in ref. [3]. Indeed, the absence of certain finite-size corrections was suggested in ref. [4] as a possible explanation for the discrepancy. In ref. [6] the finite-size data were directly compared to the renormalization prediction for $L = \infty$; no actual extrapolations to infinite system size were made. Hence, our results now confirm the renormalization prediction for the Binder cumulant of the five-dimensional Ising model for the

TABLE II. – *Results of the least-squares fits of the universal amplitude ratio Q. The numbers in parentheses denote the errors in the last digit.*

Analysis	y_t^*	Q	K_c
1	2.46 (9)	0.456 (6)	0.1139149 (7)
2	2.50 (fixed)	0.454 (5)	0.1139147 (6)
3	2.50 (fixed)	0.45694658... (fixed)	0.1139150 (4)

first time, in the sense that the accuracy of our analysis exceeds the level needed to distinguish between the competing results for Q [3], [10].

Because the thermal exponent agrees with the predicted value, we have repeated the analysis with y_t^* fixed at this value. The resulting estimate for Q again agrees with the prediction. However, comparing the uncertainty in Q with the error margins quoted in ref. [4], where y_t^* was also kept fixed, we see that the results for Q for the systems with long-range interactions are even more accurate. Given the large amount of CPU-time spent on the five-dimensional case, this illustrates how well suited the low-dimensional long-range systems are for the study of universal properties above the upper critical dimension. Finally, in order to lower the uncertainty in K_c, we have made a third analysis *assuming* that Q takes its theoretical value. All three estimates for the critical coupling agree within one standard deviation.

In order to gain some insight into the nature of the finite-size corrections affecting Q, we have studied $Q(K_c, L)$ as a function of L [13]. Most of our data were taken at $K = 0.1139100$, slightly different from our best estimate for K_c. Therefore we have corrected these data for the difference in coupling strength using eq. (3). Figure 1 shows both $Q(K = 0.1139100, L)$ and $Q(K_c, L)$ as a function of L. This turns out to be a surprisingly instructive plot. Firstly, one notices that for the larger values of L the finite-size data for Q are *strongly* dependent on the coupling, which is due to the large value of y_t^*. This implies that an incorrect estimate of K_c has a considerable effect on the resulting estimate of the Binder cumulant. Secondly, one observes that the dashed curve indicating the finite-size corrections as predicted by renormalization theory gives a good description of the data down to system sizes as small as 4 or 5 (cf. refs. [5], [7], [8]). The overall approach to the $L \to \infty$ limit is very slow, given the huge number of spins in the largest system. Returning to the original discrepancy, we have repeated the least-squares fits with a smaller number of correction terms. Apart from an increase in χ^2, indicating the importance of the higher-order terms, this leads to a higher estimate of the critical coupling and a correspondingly higher value for Q, although it was by no means as high as the result in [3]. On the other hand, the shift term $\propto L^{-1/2}$ is not the dominant term for small L, as already suggested in refs. [4], [7], but it is neither negligibly small (in contrast with the results for systems with long-range interactions). Naturally, for large L it will dominate all other corrections.

Unlike the Binder cumulant, the critical coupling was estimated in many studies. Let us therefore compare our estimate for K_c with these previous estimates (table III). The early result by Fisher and Gaunt [14] already has a remarkable accuracy, but the quoted uncertainty turns out to be almost ten times too small. Other series expansion results [15], [16] agree with our prediction; in particular the result of Guttmann (which was obtained by fixing the critical exponent γ at its mean-field value). Still, the uncertainty in this estimate is more than an order of magnitude larger than in the newest Monte Carlo result. The most accurate result until now from equilibrium Monte Carlo simulations was obtained by Parisi and Ruiz-Lorenzo [6] and lies

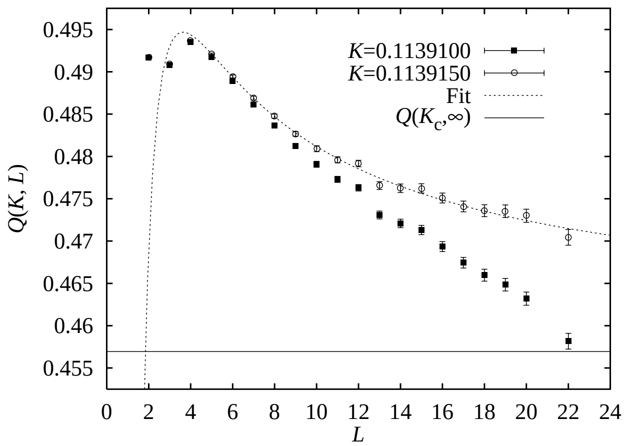

Fig. 1. – The Binder cumulant at $K = 0.1139100$, where most of our data were taken, and $K = 0.1139150$, our best estimate of K_c, *vs.* the system size L. The points at the latter coupling were calculated from those at the former coupling. Furthermore the function describing the finite-size corrections at criticality (dashed curve) and the $L \to \infty$ limit of the Binder cumulant (solid line) are shown.

TABLE III. – *Critical couplings for the five-dimensional Ising model as obtained in various studies.*

Reference	Year	K_c	Method	Remarks
[14]	1964	0.114035 (13)	series exp.	
[15]	1981	0.113917 (7)	series exp.	γ fixed
[1], [2]	1985	0.1140	Monte Carlo	$L \leq 7$
[16]	1993	0.113935 (15)	series exp.	
[16]	1993	0.11391 (1)	dynamic MC	$L \leq 48$
[3]	1994	0.113929 (45)	Monte Carlo	$L \leq 17$
[5]	1996	0.11389 (13)	Monte Carlo	$L \leq 14$
[6]	1996	0.11388 (3)	Monte Carlo	$L \leq 16$, y_t^* fixed
[17]	1996	0.11391 (1)	dynamic MC	$L = 112$
This work	1997	0.1139149 (7)	Monte Carlo	$L \leq 22$
This work	1997	0.1139150 (4)	Monte Carlo	$L \leq 22$, Q and y_t^* fixed

one σ below our estimate. Since this value was obtained with y_t^* fixed, the uncertainty has to be compared to that of the second analysis in table II. Finally, two (coinciding) estimates were obtained by studying the critical dynamics of the five-dimensional Ising model [16], [17] for very large system sizes and requiring that the effective dynamical critical exponent approaches its asymptotic value $z = 2$. These results are also in good agreement with our estimate and the latter may hence be used to make a more accurate study of the critical dynamics of the five-dimensional Ising model.

Conclusion. – In this paper we have presented numerical results for the five-dimensional Ising model, in particular for the Binder cumulant and the critical coupling. Using accurate results for relatively large system sizes we have been able to carry out a detailed analysis of the various corrections to scaling. The results are in full agreement with the predictions of renormalization theory and hence resolve the long-standing discrepancy for the Binder cumulant. Furthermore, we reinforced our earlier suggestion that this discrepancy was caused by the neglect of higher-order finite-size corrections. A more elaborate analysis of the critical properties of the five-dimensional Ising model will be presented elsewhere.

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